# Extending hyperelliptic K3 surfaces, and Godeaux surfaces with torsion $\mathbb{Z}/2$

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**Abstract**

We study the extension of a hyperelliptic K3 surface to a Fano 6-fold. This determines a family of surfaces of general type with $p_g = 1$, $K^2 = 2$ and hyperelliptic canonical curve, where each surface is a weighted complete intersection inside a Fano 6-fold. Finally, we use these hyperelliptic surfaces to determine an 8-parameter family of Godeaux surfaces with torsion $\mathbb{Z}/2$.

## 1 Introduction

We begin by studying extensions of certain hyperelliptic K3 surfaces. These surfaces are the hyperelliptic degeneration of the symmetric determinantal quartic surfaces studied in [Co2]. Let $D$ be a hyperelliptic curve of genus 3 and $A$ be an ineffective theta characteristic on $D$. We study extensions of the graded ring

$$R(D, A) = \bigoplus_{n \geq 0} H^0(D, \mathcal{O}_D(nA))$$

with $\operatorname{Proj} R(D, A)$ defining $D \subset \mathbb{P}(2^3, 3^4, 4)$. In the first instance, there is an extension of $D$ to a hyperelliptic K3 surface $T \subset \mathbb{P}(2^4, 3^4, 4)$ with $10 \times \frac{1}{2}(1, 1)$ points and containing $D$ as a weighted hyperplane section of degree 2. Sections 2 and 3 of this paper treat graded rings over hyperelliptic curves and K3 surfaces respectively, by working relative to the hyperelliptic double covering.

Now, the K3 surface $T$ is the elephant hyperplane section of a Fano 3-fold $W \subset \mathbb{P}(1, 2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points. We think of this 3-fold as an extension of $T$, and make further extensions up to a Fano 6-fold $W^6$, with each successive $W_i$ containing $T$ as an appropriate number of hyperplane sections. This leads to the tower

$$D \subset T \subset W^3 \subset W^4 \subset W^5 \subset W^6 \subset \mathbb{P}(1^4, 2^4, 3^4, 4).$$

This is the hyperelliptic degeneration of the symmetric determinantal quartic extensions constructed in [Co2]. Thus we can hope for a one-to-one correspondence of moduli between the surface $T$ and the 6-fold $W^6$. Indeed, we have

**Main Theorem 1.1** *For each quasismooth hyperelliptic K3 surface $T \subset \mathbb{P}(2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points, there is a unique extension to a quasismooth Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points and such that*

$$T = W \cap H_1 \cap H_2 \cap H_3 \cap H_4,$$

*where the $H_i$ are hyperplanes of the projective space $\mathbb{P}(1^4, 2^4, 3^4, 4)$.*

The Main Theorem is proved in Section 4, using a projection–unprojection construction for $T$ and for $W^6$. See [PR] and [R3] for details on projection–unprojection methods.

We call the Fano 6-fold $W$ a *key variety*, because it contains several interesting varieties as appropriate weighted complete intersections. We already know how to recover the curve $D$ and the K3 surface $T$ from $W^6$, and we can also construct hyperelliptic surfaces of general type.

**Corollary 1.2** *There is a 15-parameter family of hyperelliptic surfaces $Y$ of general type with $p_g = 1$, $q = 0$, $K^2 = 2$ and no torsion, each of which is a complete intersection of type $(1, 1, 1, 2)$ in a Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points.*

Each surface $Y$ contains the genus 3 hyperelliptic curve $D \in |K_Y|$ as the unique hyperplane section of weight 1. These surfaces were constructed by Catanese and Debarre in [CD], but our method has the advantage of being more widely applicable to other situations.

In Section 5 we use the key variety method to construct a new family of Godeaux surfaces with torsion $\mathbb{Z}/2$.

**Theorem 1.3** *There is an 8-parameter family of Godeaux surfaces $X$ with torsion $\mathbb{Z}/2$, where each $X$ is obtained as a $\mathbb{Z}/2$-quotient of some hyperelliptic surface $Y$ constructed in Corollary 1.2.*

The Godeaux surfaces are surfaces of general type with $p_g = 0$, $K^2 = 1$, and their torsion group is cyclic of order $\leq 5$. The components of the moduli space with torsion $\mathbb{Z}/5$, $\mathbb{Z}/4$, $\mathbb{Z}/3$ were constructed in [R1], and in each case the moduli space is irreducible, unirational, and 8-dimensional, which is the expected dimension. The first simply connected example appeared in [B], and recently another simply connected Godeaux surface was constructed in [LP] using $\mathbb{Q}$-Gorenstein smoothing theory. It is expected that the key variety method will give an irreducible 8-dimensional component of the moduli space of Godeaux surfaces with algebraic fundamental group $\mathbb{Z}/2$, although we have not proved that here.

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## 2 Graded rings over hyperelliptic curves

In this section we review hyperelliptic curves and their graded rings, using the double covering of $\mathbb{P}^1$. This is well known material covered in section 4 of [R2]. We include this section for completeness since we will generalise to hyperelliptic K3 surfaces in Section 3.

We consider the case when $D$ is a hyperelliptic curve of genus 3. Then the canonical linear system $|K_D|$ defines a double covering of $\mathbb{P}^1$ embedded as a plane conic, branched in 8 points $Q_1, \dots, Q_8$. The corresponding ramification (or Weierstrass) points on $D$ are labelled $P_1, \dots, P_8$. The double covering $\pi\colon D \to \mathbb{P}^1$ determines and is determined by the $g^1_2$: a free linear system of dimension 1 and degree 2. Moreover, we have $2P_i \sim g^1_2$ and $K_D \sim 2g^1_2$. There is a natural hyperelliptic involution $h$ on $D$ which swaps the two sheets of the double covering, and $\pi$ is the quotient map of this involution.

Choose generators $s_1$, $s_2$ of $H^0(D, g^1_2)$. These are coordinates on $\mathbb{P}^1$, and there is a polynomial $F_8(s_1, s_2)$ whose vanishing determines the branch locus $Q_1 + \cdots + Q_8$ of $\pi$. The double covering is $D_8 \subset \mathbb{P}(1, 1, 4)$, defined by the equation $w^2 = F_8(s_1, s_2)$. By considering rational functions on $D$, we have

$$4g^1_2 \sim P_1 + \cdots + P_8,$$

or more generally,

$$P_1 + \cdots + P_a + (8 - a)g^1_2 \sim P_{a+1} + \cdots + P_8 + 4g^1_2.$$

We write $B_1 = P_1 + \cdots + P_a$, $B_2 = P_{a+1} + \cdots + P_8$, then since the $B_i$ are effective Cartier divisors on $D$ we can choose constant sections

$$u\colon \mathcal{O}_D \to \mathcal{O}_D(B_1), \quad v\colon \mathcal{O}_D \to \mathcal{O}_D(B_2).$$

Now $u^2, uv, v^2$ are sections of $ag^1_2$, $4g^1_2$, $(8 - a)g^1_2$ respectively, so we have two relations $u^2 = f(s_1, s_2)$, $v^2 = g(s_1, s_2)$ and the identity $w = uv$. Here $f(s_1, s_2)$ is a homogeneous function of degree $a$ on $\mathbb{P}^1$ with zeros at $Q_1, \dots, Q_a$, similarly $g(s_1, s_2)$, so that $F = fg$.

Clearly every $h$-invariant divisor class can be written in the form

$$A \sim P_1 + \cdots + P_a + bg^1_2 \sim P_{a+1} + \cdots + P_8 + (a + b - 4)g^1_2.$$

For such a divisor $A$, the graded ring

$$R(D, A) = \bigoplus_{n \geq 0} H^0(D, \mathcal{O}_D(nA))$$

can be studied relative to the base $\mathbb{P}^1$ via the double covering $\pi$. We quote the following proposition from [R2] for $D$ of genus 3, although the proposition and subsequent graded ring calculations work for any genus with only minor alterations.

**Proposition 2.1** *Let $D$ be a hyperelliptic curve of genus 3 with Weierstrass points $P_1, \dots, P_8$, and write $\pi\colon D \to \mathbb{P}^1$ for the natural quotient by the hyperelliptic involution $h$. Then*

*(1)* $\pi_*\mathcal{O}_D = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4)$;

*(2)* $\pi_*\mathcal{O}_D(g^1_2) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$;

*(3)* $\pi_*\mathcal{O}_D(P_1 + \cdots + P_a) = \mathcal{O}_{\mathbb{P}^1}u \oplus \mathcal{O}_{\mathbb{P}^1}(a-4)v$;

*where in each case the first summand is invariant under $h$ and the second is anti-invariant.*

**Remark 2.2** Note that in case (1) the direct image sheaf is a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras, where the multiplication

$$\mathcal{O}_{\mathbb{P}^1}(-4) \otimes \mathcal{O}_{\mathbb{P}^1}(-4) \to \mathcal{O}_{\mathbb{P}^1}$$

is defined via $w^2 = F(s_1, s_2)$.

## 2.1 The ineffective theta characteristic

Consider the ineffective theta characteristic

$$A \sim P_1 + \cdots + P_4 - g^1_2 \sim P_5 + \cdots + P_8 - g^1_2$$

on $D$. An ineffective theta characteristic is a divisor class on $D$ such that $h^0(D, A) = 0$ and $2A \sim K_D$. Using Proposition 2.1, we see that $R(D, A)$ is generated by monomials in $s_1, s_2, u, v$:

| $n$ | $H^0(D, \mathcal{O}_D(nA))$ | $H^0(\mathbb{P}^1, \pi_*\mathcal{O}_D(nA))$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | $\phi$ | $\phi$ |
| 2 | $y_1,\ y_2,\ y_3$ | $s_1^2,\ s_1s_2,\ s_2^2$ |
| 3 | $z_1,\ z_2,\ z_3,\ z_4$ | $s_1u,\ s_2u,\ s_1v,\ s_2v$ |
| 4 | $t$ | $uv$ |

The relations between these generators are either of the trivial monomial kind, or derived from

$$u^2 = f_4(s_1, s_2), \ v^2 = g_4(s_1, s_2).$$

For example, it is clear that $z_1^2 = s_1^2 u^2 = y_1 f(y_1, y_2, y_3)$, where $f(y_1, y_2, y_3)$ is a *rendering* of $f_4(s_1, s_2)$ in the quadratic monomials $s_1^2$, $s_1 s_2$, $s_2^2$. In fact, we can present all the equations as

$$\operatorname{rank}\left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & f_2 & t \\ z_3 & z_4 & t & g_2 \end{array}\right) \leq 1,$$

where $f_2$ and $g_2$ are quadrics in $y_1$, $y_2$, $y_3$. Taking $\operatorname{Proj} R(D, A)$ gives

$$D \subset \mathbb{P}(2^3, 3^4, 4),$$

and the double covering of $\mathbb{P}^1$ is the conic defined by the first $2 \times 2$ minor of the matrix.

# 3 Graded rings over hyperelliptic K3 surfaces

In this section we generalise the methods of Section 2 to hyperelliptic K3 surfaces, and in 3.1 we construct a hyperelliptic K3 surface $T$ that extends the hyperelliptic curve $D$ of 2.1. Section 3.2 gives alternative descriptions of $D$ and $T$, while 3.3 describes a projection construction for $T$ which will be used in the proof of the Main Theorem 1.1.

A hyperelliptic K3 surface $T$ is a K3 surface together with a complete linear system $L$ such that $|L|$ contains an irreducible hyperelliptic curve $D$ of arithmetic genus $g = h^0(T, \mathcal{O}_T(L)) - 1$. Then $L$ determines a 2-to-1 map $\pi \colon T \to F$ where $F$ is a surface of degree $g - 1$ in $\mathbb{P}^g$. The branch locus of the double covering is some divisor in $|-2K_F|$. For further details on the hyperelliptic dichotomy for K3 surfaces see [SD]. Del Pezzo classified the possibilities for $F$ as rational scrolls or the Veronese surface. Since both have very simple explicit descriptions, we can analyse graded rings over any hyperelliptic K3 surface by calculating relative to the base $F$. For brevity we treat only the case $g = 3$, but more general examples are contained in [Co1].

We assume that $F = Q_2 \subset \mathbb{P}^3$ is a quadric of rank 4 and the double cover $\pi \colon T \to F$ is branched in a curve $C$ of bidegree $(4, 4)$, which splits into two components $C_1 + C_2$ of bidegree $(3, 1)$ and $(1, 3)$ respectively. The components of the branch curve intersect one another transversally in 10 points which are nodes of $T$. This is the hyperelliptic degeneration of the symmetric determinantal quartic K3 surface of [Co2]. As usual there is a hyperelliptic involution $h \colon T \to T$ exchanging the two sheets of the double cover, and $\pi$ is the quotient map of $h$. Let $H_1$, $H_2$ be the generators of $\operatorname{Pic} Q$, then we omit $\pi^*$ to write $\pi^* H_i = H_i$ on $T$, and $\pi^* C_i = 2D_i$.

Let $s_1, s_2$ be generators of $H^0(T, H_1)$, similarly $t_1, t_2$ for $H^0(T, H_2)$. Then there is an equation $F_{4,4}(s_1, s_2, t_1, t_2)$ defining the branch curve $C$ on $Q$. This equation factors as $F = f_{3,1}(s_i, t_i) g_{1,3}(s_i, t_i)$, which determines the splitting $C = C_1 + C_2$. The double cover $T$ is given by $w^2 = F$, and we have $2D_1 \sim 3H_1 + H_2$ and $2D_2 \sim H_1 + 3H_2$ on $T$. Considering the rational function $w/(t_1^2 s_1^2)$ on $T$, we find

$$2(H_1 + H_2) \sim D_1 + D_2.$$

By analogy with the hyperelliptic curves of section 2 we write down graded rings

$$R(T, A) = \bigoplus_{n \geq 0} H^0(T, \mathcal{O}_T(nA))$$

where $A$ is a divisor class which is invariant under $h$. Any such $A$ can be written in the form

$$A \sim D_1 + n_1 H_1 + n_2 H_2 \sim D_2 + (n_1 + 1) H_1 + (n_2 - 1) H_2.$$

The following proposition is a natural extension of Proposition 2.1, which allows us to describe $R(T, A)$ relative to $R(Q, \pi_* A)$.

**Proposition 3.1** *Let $T$ be a hyperelliptic K3 surface double covering of the rank 4 quadric $Q \subset \mathbb{P}^3$, with ramification properties as described above. Choose constant sections $u \colon \mathcal{O}_T \to \mathcal{O}_T(D_1)$ and $v \colon \mathcal{O}_T \to \mathcal{O}_T(D_2)$ for the components $D_i$ of the ramification curve. Clearly we have $u^2 = f_{3,1}(s_i, t_i)$, $uv = w$ and $v^2 = g_{1,3}(s_i, t_i)$, where $F = fg$. Moreover,*

*(1)* $\pi_* \mathcal{O}_T = \mathcal{O}_Q \oplus \mathcal{O}_Q(-2, -2)$;

*(2)* $\pi_* \mathcal{O}_T(H_1) = \mathcal{O}_Q(1, 0) \oplus \mathcal{O}_Q(-1, -2)$;

*(3)* $\pi_* \mathcal{O}_T(D_1) = \mathcal{O}_Q u \oplus \mathcal{O}_Q(1, -1) v$;

*with similar results for $H_2$, $D_2$ respectively.*

**Remark 3.2** Once again we note the $\mathcal{O}_Q$-algebra structure on $\pi_* \mathcal{O}_T$. The multiplication map

$$\mathcal{O}_Q(-2, -2) \otimes \mathcal{O}_Q(-2, -2) \to \mathcal{O}_Q$$

is defined via the equation $w^2 = F_{4,4}(s_1, s_2, t_1, t_2)$.

### 3.1 Construction of the K3 surface $T$

Write $A \sim D_1 - H_1 \sim D_2 - H_2$, which is an $h$-invariant divisor class on $T$, satisfying $H^0(T, \mathcal{O}_T(A)) = 0$ and $\mathcal{O}_T(A)^{[2]} = \pi^*\mathcal{O}_Q(1)$. Note that $A$ is the analogue of the ineffective theta characteristic in Section 2.1. We can describe the ring $R(T, A)$ using Proposition 3.1. The generators for $R(T, A)$ are:

| $n$ | $H^0(T, \mathcal{O}_T(nA)$ | $H^0(Q, \pi_*\mathcal{O}_T(nA))$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 2 | $y_1,\ y_2,\ y_3,\ y_4$ | $s_1t_1,\ s_2t_1,\ s_1t_2,\ s_2t_2$ |
| 3 | $z_1,\ z_2,\ z_3,\ z_4$ | $t_1u,\ t_2u,\ s_1v,\ s_2v$ |
| 4 | $t$ | $uv = w$ |

The relations are again mostly trivial monomial relations, together with those derived from $u^2 = f_{3,1}$ and $v^2 = g_{1,3}$. Some are slightly more difficult to write down than others, for example,

$$z_1t = t_1u^2v = t_1vf_{3,1} = s_1vq_{2,2} + s_2vq'_{2,2} = z_3q_1(y_i) + z_4q'_1(y_i),$$

where $q_1$ and $q'_1$ are suitable quadrics rendered in $y_1$,. . . ,$y_4$. The trick here is to make $f_{3,1}$ bihomogeneous by incorporating the factor $t_1$ into $f$ and simultaneously taking out the excess in $s_1$, $s_2$. Clearly we can not expect $f$ to be divisible by $s_1$ or by $s_2$, and so we have a choice of ways to break up $f$ into quadrics. Fortunately, this choice is arbitrary, as any discrepancy is accounted for by the rank condition (1) below. We present all the relations of $R(T, A)$ as follows

$$\operatorname{rank}\left(\begin{array}{cc|c} y_1 & y_2 & z_1 \\ y_3 & y_4 & z_2 \\ \hline z_3 & z_4 & t \end{array}\right) \leq 1, \tag{1}$$

$$\begin{aligned} z_1^2 &= t_1^2f_{3,1} & z_3^2 &= s_1^2g_{1,3} \\ z_1z_2 &= t_1t_2f_{3,1} & z_3z_4 &= s_1s_2g_{1,3} \\ z_2^2 &= t_2^2f_{3,1} & z_4^2 &= s_2^2g_{1,3} \\ z_1t &= q_1z_3 + q'_1z_4 & z_3t &= q_3z_1 + q'_3z_2 \\ z_2t &= q_2z_3 + q'_2z_4 & z_4t &= q_4z_1 + q'_4z_2 \end{aligned}$$

$$t^2 = F(y_i),$$

where for example $z_1^2 = t_1^2f_{3,1}$ means we render the bihomogeneous expression $t_1^2f_{3,1}$ in the variables $y_1, \ldots, y_4$. Then $\operatorname{Proj} R(T, A)$ gives us the K3 surface

$$T \subset \mathbb{P}(2^4, 3^4, 4),$$

which has $10 \times \frac{1}{2}(1,1)$ points. Note that the curve $D$ of Section 2.1 is obtained by taking a hyperplane section of weight 2 in $T$, avoiding the $\frac{1}{2}$ points.

### 3.2 Alternative descriptions of hyperelliptic varieties

We can consider the curve $D$ as a codimension 2 complete intersection inside a weighted homogeneous variety as follows: let $X$ be the second Veronese embedding of $\mathbb{P}^3$ with coordinates $s_1, s_2, u, v$, and take the affine cone $\mathcal{C}X \subset \mathbb{A}^{10}$ over $X$. Aside from the obvious $\mathbb{C}^\times$-action on $\mathcal{C}X$ there are many other possibilities, and we choose a weighted $\mathbb{C}^\times$-action with weights $(1,1,2,2)$. Then the quotient $Y = \mathcal{C}X /\!\!/_1 \mathbb{C}^\times$ of $\mathcal{C}X$ is contained in $\mathbb{P}(2^3, 3^4, 4^3)$, and is defined by the equations

$$\operatorname{rank}\left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & x_1 & t \\ z_3 & z_4 & t & x_2 \end{array}\right) \leq 1.$$

The hyperelliptic curve $D$ is simply the codimension 2 complete intersection $x_1 = f_2$, $x_2 = g_2$ inside $Y$.

Similarly, $T$ is a codimension 2 complete intersection in the weighted homogeneous space we now describe. Consider the following $(\mathbb{C}^\times)^2$-action on $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$:

$$\begin{aligned} \lambda &\colon (s_1, s_2, t_1, t_2, u, v) \mapsto (\lambda^2 s_1, \lambda^2 s_2, t_1, t_2, \lambda^3 u, \lambda v) \\ \mu &\colon (s_1, s_2, t_1, t_2, u, v) \mapsto (s_1, s_2, \mu^2 t_1, \mu^2 t_2, \mu u, \mu^3 v). \end{aligned}$$

The 4-dimensional quotient $Z = (\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2) /\!\!/_{(1,1)} (\mathbb{C}^\times)^2$ is embedded in $\mathbb{P}(2^4, 3^4, 4)$ by the determinantal equations (1). The surface $T$ is the complete intersection $u^2 = f_{3,1} \in (6,2)$, $v^2 = g_{1,3} \in (2,6)$ in $Z$.

### 3.3 Projection of $T$ to a complete intersection

Consider the following projection map on del Pezzo surfaces: let $Q \subset \mathbb{P}^3$ be a quadric of rank 4 and blow up a point $P$ on $Q$ to obtain the del Pezzo surface $B$. Then contract the two $(-1)$-curves on $B$ arising from the rulings of $Q$ to get $\mathbb{P}^2$. Now, suppose we have a curve $C$ on $Q$ of type $(4,4)$ which splits as $C = C_1 + C_2$ where $C_1 \in (3,1)$, $C_2 \in (1,3)$ so that $C$ has 10 nodes. If the centre of projection $P$ is chosen to be one of these nodes then the two components $C_1$, $C_2$ are projected to nodal plane cubics, and the image of $P$ is the line $L$ through these two nodes.

Now suppose we have a hyperelliptic $K3$ surface $T$ which is a double cover of $Q$ branched in $C$. The classical projection $Q \dashrightarrow \mathbb{P}^2$ lifts to the

double cover as illustrated by the diagram below:

$$\begin{tikzcd} & \widetilde{T} \arrow[dl, "\sigma"'] \arrow[d] \arrow[dr, "\pi"] & \\ T \arrow[d, "2\ to\ 1"'] & B \arrow[dl] \arrow[dr] & T' \arrow[d, "2\ to\ 1"] \\ Q & & \mathbb{P}^2 \end{tikzcd}$$

where $\sigma\colon \widetilde{T} \to T$ is the blowup of $P$ in $T$ and we write $E \cong \mathbb{P}^1$ for the exceptional divisor. The image $T'$ of the projection is a double cover of $\mathbb{P}^2$ branched over the two nodal cubics. The centre of projection $P$ in $T$ is projected to a rational curve of arithmetic genus 2 double covering $L$ away from the two nodes, and branched over the residual intersection with $C$.

Now this diagram can also be recast as a projection–unprojection operation in the sense of [R3], [PR]. Start from $T \subset \mathbb{P}(2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points and polarising divisor $A$, as described in section 3.1. Choose a $\frac{1}{2}$ point $P$ in $T$, and write $\sigma\colon \widetilde{T} \to T$ for the $(1,1)$-weighted blowup of $P$, whose exceptional curve is $E$. Then the projected surface $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ is calculated as

$$T' = \operatorname{Proj} R(\widetilde{T}, \sigma^* A - \tfrac{1}{2}E),$$

where certain functions on $\widetilde{T}$ are eliminated by the projection since they do not vanish appropriately along $E$. Geometrically, the surface $T'_{6,6}$ is a double covering of $\mathbb{P}^2$ branched in the two nodal cubics defined by the equations of degree 6.

This projection to $T'$ can be expressed explicitly as an operation in commutative algebra. Assume the centre of projection is a $\frac{1}{2}$ point at the coordinate point $P_{y_4}$, with local coordinates $z_3, z_4$. Then adjusting the notation of section 3.1 slightly, write down the matrix relations

$$\operatorname{rank} \begin{pmatrix} y_2 & f & z_1 \\ g & y_4 & z_3 \\ z_2 & z_4 & t \end{pmatrix} \le 1, \tag{2}$$

where we reserve the right to choose $f$, $g$ later. These equations are a subset of those for $T \subset \mathbb{P}(2^4, 3^4, 4)$ after a trivial change of coordinates. The remaining equations for $T$ are completely determined by

$$\begin{aligned} z_1^2 &= L_1 y_2^2 + L_2 y_2 f + L_3 f^2 \\ z_2^2 &= M_1 y_2^2 + M_2 y_2 g + M_3 g^2 \end{aligned}$$

where a priori $L_i$, $M_i$ are linear in $y_1, \dots, y_4$. Indeed, the equations we have written down so far are sufficient to determine the two components of the

branch curve, and their defining equations $f_{3,1}$ and $g_{1,3}$. We can fill in the remaining equations of $T$ using the procedure outlined in Section 3.1.

Since we fixed a $\frac{1}{2}$ point at $P_{y_4}$, the last equation for $T$ can be written as

$$t^2 = a_2(y_1, y_3)y_4^2 + b_3(y_1, y_2, y_3)y_4 + c_4(y_1, y_2, y_3).$$

Now the tangent cone to $P$ must factorise because the branch curve $C$ splits into two components, so we can choose coordinates

$$f = y_1 + \alpha y_3, \quad g = \beta y_1 + y_3$$

so that $a = y_1 y_3$. This in turn forces $L_3 = y_1$, $M_3 = y_3$ so that modulo the minors of matrix (2), the equations involving $z_1^2$ and $z_2^2$ take the form

$$\begin{aligned} z_1^2 &= L_1(y_1, y_2, y_3)y_2^2 + l_4 y_2 f g + y_1 f^2 \\ z_2^2 &= M_1(y_1, y_2, y_3)y_2^2 + m_4 y_2 f g + y_3 g^2, \end{aligned} \tag{3}$$

where $L_1$, $M_1$ do not involve $y_4$ and $l_4$, $m_4$ are scalars. The image of the exceptional curve $E$ is defined by $y_2 = 0$.

We are finally in a position to describe the projection centred at $P_{y_4}$ in terms of explicit equations. The local coordinates near $P$ are $z_3$, $z_4$ so we expect the projection to eliminate these variables along with $y_4$ (see [R3], example 9.13). In fact the projection also eliminates $t$, and we are left with equations (3) defining a complete intersection

$$T'_{6,6} \subset \mathbb{P}(2, 2, 2, 3, 3).$$

This is the hyperelliptic degeneration of the *totally tangent conic* configuration of [Co2].

# 4 Extending hyperelliptic graded rings

In this section we consider extensions of the hyperelliptic K3 surface $T$ constructed in section 3.1, and prove the Main Theorem 1.1. As with the symmetric determinantal extensions of [Co2], the most convenient way to extend the K3 surface $T$ is by using the projection construction of Section 3.3. We start from

$$\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2, 2, 2, 3, 3),$$

where $T'$ is a double covering of $\mathbb{P}(2, 2, 2)$ branched in two nodal cubics. The image of $\varphi$ is a curve of arithmetic genus 2, which is a double cover of the line joining the two nodes. Constructing $\varphi$ and $T'_{6,6}$ is equivalent to constructing $T$ itself, so we prove the theorem by extending $\varphi$ and $T'$.

We assume that $\varphi$ is a double cover of the line $(y_2 = 0) \subset \mathbb{P}(2, 2, 2)$ branched over the points $\varphi(1, 0)$ and $\varphi(0, 1)$. Then for general $T'$ the map $\varphi$ is

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}(2, 2, 2, 3, 3)$$

$$(u,v) \mapsto \left(u^2, 0, v^2, u(u^2+\alpha v^2), v(\beta u^2+v^2)\right). \tag{4}$$

Rendering $\varphi^*(z_i^2)$ in terms of $y_1$, $y_3$ we see that the image of $\varphi$ is defined by the equations

$$C_1\colon z_1^2 = y_1(y_1+\alpha y_3)^2 \tag{5}$$
$$C_2\colon z_2^2 = y_3(\beta y_1+y_3)^2 \tag{6}$$
$$y_2 = 0. \tag{7}$$

To define $T' \subset \mathbb{P}(2,2,2,3,3)$ we must choose two appropriate combinations of weight 6 in equations (5–7). Note that if we want the branch curves to be nondegenerate then we should ensure that both equations for $T'$ involve $y_2$ nontrivially. Moreover, after incorporating $y_2$ into the equations we should check that there are still two bona fide nodes on the branch locus at $(-\alpha,0,1)$ and $(1,0,-\beta)$. So, calculating the tangent cone to each curve at these points forces the equations of $T'$ to take the form

$$\begin{array}{l} C_1 + l_1Q_1 + l_2Q_2 + l_3Q_3 + l_4Q_4 \\ C_2 + m_1Q_1 + m_2Q_2 + m_3Q_3 + m_4Q_4, \end{array} \tag{8}$$

where $\alpha$, $\beta$, $l_i$, $m_i$ are scalar parameters and

$$Q_1 = (y_1+\alpha y_3)y_2^2, \quad Q_2 = y_2^3, \quad Q_3 = (\beta y_1 + y_3)y_2^2,$$
$$Q_4 = (y_1+\alpha y_3)(\beta y_1+y_3)y_2.$$

**Proof of main theorem** The proof follows a similar approach to the main result of [Co2] and it is informative to compare the two at each stage. We explicitly extend the projected image $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ to a Fano 6-fold $W'_{6,6} \subset \mathbb{P}(1^4,2^3,3^2)$ containing the image of $\mathbb{P}^5$ under some map $\Phi$. Define $\varphi\colon \mathbb{P}^1 \to \mathbb{P}(2,2,2,3,3)$ as in (4) and write $\varphi_0\colon \mathbb{P}^1 \to \mathbb{P}(2,2,2)$ for the map

$$\varphi_0^*(y_1) = u^2, \quad \varphi_0^*(y_2) = 0, \quad \varphi_0^*(y_3) = v^2.$$

Then writing $u$, $v$, $a$, $b$, $c$, $d$ for the coordinates on $\mathbb{P}^5$, up to automorphisms of $\mathbb{P}^5$ and $\mathbb{P}(1^4,2^3)$ the general extension of $\varphi_0$ to $\Phi_0\colon \mathbb{P}^5 \to \mathbb{P}(1^4,2^3)$ is

$$\Phi_0^*(a) = a, \quad \Phi_0^*(b) = b, \quad \Phi_0^*(c) = c, \quad \Phi_0^*(d) = d,$$

$$\begin{array}{ll} \Phi_0^*(y_1) = u^2 & +2av, \\ \Phi_0^*(y_2) = 0 + & bu + cv, \\ \Phi_0^*(y_3) = v^2 + 2du & \end{array} \tag{9}$$

We prove that there is a unique map $\Phi\colon \mathbb{P}^5 \to \mathbb{P}(1^4,2^4,3^2)$ which is a lift of $\Phi_0$ and which extends $T'_{6,6}$ to $W'_{6,6}$.

Write $M$, $R$, $S$ for the coordinate rings of $\mathbb{P}^5$, $\mathbb{P}(1^4, 2^3)$ and $\mathbb{P}(1^4, 2^3, 3^2)$ respectively. By equation (9), the map $\Phi_0^*$ induces a graded $R$-module structure on $M$ with generators 1, $u$, $v$ and $uv$. Similarly $\Phi^*$ makes $M$ into a graded $S$-module with the same generators. The presentation of $M$ as a module over $R$ is

$$0 \leftarrow M \xleftarrow{(1,u,v,uv)} R \oplus 2R(-1) \oplus R(-2) \xleftarrow{A} R(-2) \oplus 2R(-3) \oplus R(-4)$$

where $A$ is the matrix

$$\begin{pmatrix} -y_2 & by_1 & cy_3 & -2cdy_1 + 4ady_2 - 2aby_3 \\ b & -y_2 & -2cd & cy_3 \\ c & -2ab & -y_2 & by_1 \\ 0 & c & b & -y_2 \end{pmatrix}. \tag{10}$$

Since $\Phi$ is a lift of $\varphi$ we assume that the general forms of $\Phi^*(z_i)$ are

$$\begin{aligned} \Phi^*(z_1) &= u^3 + \alpha uv^2 + s_1u^2 + s_2uv + s_3v^2 + s_4u + s_5v \\ \Phi^*(z_2) &= \beta u^2v + v^3 + t_1u^2 + t_2uv + t_3v^2 + t_4u + t_5v \end{aligned}$$

where the $s_i(a, b, c, d)$, $t_i(a, b, c, d)$ are homogeneous polynomials of degree 1 or 2 as appropriate. Then using the $R$-module structure of $M$ we can write

$$\begin{aligned} \Phi^*(z_1) &= u(f + s_4) + s_2uv + s_5v \\ \Phi^*(z_2) &= v(g + t_5) + t_2uv + t_4u \end{aligned} \tag{11}$$

where

$$f = y_1 + \alpha y_3, \quad g = \beta y_1 + y_3.$$

We have used coordinate changes $z_1 \mapsto z_1 + s_1y_1$ and similar to absorb the values of $s_1$, $s_3$, $t_1$, $t_3$ into $z_1$, $z_2$. The following theorem shows that there are unique values of $s_i$, $t_i$ for $i = 2, 4, 5$ for which there are equations extending (5), (6). As a corollary, we prove that for these unique values of $s_i$, $t_i$, there are extensions of equations $Q_1, \dots, Q_4$.

**Theorem 4.1** *(I) The kernel of $\Phi^*\colon S \to M$ contains equations extending (5), (6) of the form*

$$\begin{aligned} z_1^2 - y_1f^2 &\in R + Rz_1 + Rz_2, \\ z_2^2 - y_3g^2 &\in R + Rz_1 + Rz_2 \end{aligned}$$

*if and only if*

$$\begin{aligned} s_2 = (1 - \alpha\beta)a, \quad s_4 = \beta a^2 + \alpha^2 d^2, \quad s_5 = \alpha(\alpha\beta - 1)ad, \\ t_2 = (1 - \alpha\beta)d, \quad t_4 = \beta(\alpha\beta - 1)ad, \quad t_5 = \beta^2a^2 + \alpha d^2. \end{aligned}$$

*(II) Given part (I), the equations are*

$$\begin{aligned} z_1^2 - y_1(f+s_4)^2 &= -4(f+s_4)s_2ay_3 - 4s_2s_5dy_1 + s_2^2y_1y_3 + s_5^2y_3 \\ &\quad + 2(1-\alpha\beta)a^2(3dz_1 - az_2) - 2\alpha a(f+s_4)z_2 \end{aligned} \tag{12}$$

$$\begin{aligned} z_2^2 - y_3(g+t_5)^2 &= -4(g+t_5)t_2dy_1 - 4t_2t_4ay_3 + t_2^2y_1y_3 + t_4^2y_1 \\ &\quad + 2(1-\alpha\beta)d^2(3az_2 - dz_1) - 2\beta d(g+t_5)z_1 \end{aligned} \tag{13}$$

**Corollary 4.2** *The kernel of $\Phi^*$ also contains (nontrivial) equations extending $Q_i$ for $i = 1, \ldots, 4$ of the form*

$$fy_2^2,\ y_2^3,\ gy_2^2,\ fgy_2 \in R + Rz_1 + Rz_2$$

*respectively.*

**Proof** The "if" part of the theorem is proved by evaluating equations (12), (13) under $\Phi^*$ with $s_i$, $t_i$ taking the values stated in the theorem. The remainder of the proof is for the "only if" part.

Using the graded module structure of $k[u,v]$ over $k[y_1,y_2,y_3]$ via $\varphi_0^*$ we write

$$\begin{aligned} \varphi^*(z_1) &= (y_1 + \alpha y_3)u \\ \varphi^*(z_2) &= (\beta y_1 + y_3)v. \end{aligned}$$

Then squaring either of these expressions and rendering $u^2$, $v^2$ as $y_1$, $y_3$ gives equations (5), (6) immediately. We attempt to do the same rendering calculation for the extended map $\Phi^*$, using

$$\begin{aligned} u^2 &= \Phi^*(y_1) - 2av \\ v^2 &= \Phi^*(y_3) - 2du. \end{aligned}$$

We can eliminate all terms involving $u^2$ or $v^2$ from $\Phi^*(z_i^2)$ to obtain

$$\begin{aligned} \Phi^*\left(z_1^2 - y_1(f+s_4)^2 + 4(f+s_4)s_2ay_3 + 4s_2s_5dy_1 - s_2^2y_1y_3 - s_5^2y_3\right) &\equiv 0 \\ \Phi^*\left(z_2^2 - y_3(g+t_5)^2 + 4(g+t_5)t_2dy_1 + 4t_2t_4ay_3 - t_2^2y_1y_3 - t_4^2y_1\right) &\equiv 0 \end{aligned}$$

modulo $(a,b,c,d)M$. The residual parts to these congruences are

$$\begin{aligned} K &= K_u u + K_v v + K_{uv} uv, \\ L &= L_u u + L_v v + L_{uv} uv \end{aligned}$$

respectively, where

$$\begin{aligned} K_u &= 8(f+s_4)s_2ad - 2s_5^2d - 2s_2^2dy_1 + 2s_2s_5y_3 \\ K_v &= -2(f+s_4)^2a + 8s_2s_5ad + 2(f+s_4)s_2y_1 - 2s_2^2ay_3 \\ K_{uv} &= 2(f+s_4)s_5 + 4s_2^2ad \end{aligned} \tag{14}$$

and

$$\begin{aligned} L_u &= -2(g+t_5)^2 d + 8t_2t_4ad + 2(g+t_5)t_2y_3 - 2t_2^2dy_1 \\ L_v &= 8(g+t_5)t_2ad - 2t_4^2a - 2t_2^2ay_3 + 2t_2t_4y_1 \\ L_{uv} &= 2(g+t_5)t_4 + 4t_2^2ad. \end{aligned} \tag{15}$$

Now $K$, $L$ are homogeneous expressions of degree 6 in $(a,b,c,d)M$, and we prove that if they are to be contained in the submodule $R+Rz_1+Rz_2 \subset M$ then $s_i$, $t_i$ must take the values stated in the theorem. From the definition of $\Phi^*(z_i)$ in (11), the submodule $R+Rz_1+Rz_2$ is the image of the composite

$$M \xleftarrow{(1,u,v,uv)} R\oplus 2R(-1)\oplus R(-2) \xleftarrow{B} R\oplus 2R(-3)\oplus R(-2)\oplus 2R(-3)\oplus R(-4)$$

where $B$ is the matrix

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & -y_2 & by_1 & cy_3 & -2cdy_1+4ady_2-2aby_3 \\ 0 & f+s_4 & t_4 & b & -y_2 & -2cd & cy_3 \\ 0 & s_5 & g+t_5 & c & -2ab & -y_2 & by_1 \\ 0 & s_2 & t_2 & 0 & c & b & -y_2 \end{array}\right).$$

The first 3 columns of $B$ are the generators 1, $z_1$, $z_2$ and the last 4 columns are the matrix $A$ from (10), which is mapped to 0 under the composite.

We seek vectors $\xi, \eta \in R\oplus 2R(-3)\oplus R(-2)\oplus 2R(-3)\oplus R(-4)$ such that

$$\begin{aligned} K &= \begin{pmatrix}1, & u, & v, & uv\end{pmatrix} B\xi, \\ L &= \begin{pmatrix}1, & u, & v, & uv\end{pmatrix} B\eta. \end{aligned} \tag{16}$$

In order to solve for $\xi$, $\eta$ and consequently fix the values of $s_i$, $t_i$ we stratify $K, L$ according to degree in $y_1, y_2, y_3$. In other words, write

$$\begin{aligned} K &= K^{(0)} + K^{(1)} + K^{(2)} \\ L &= L^{(0)} + L^{(1)} + L^{(2)} \end{aligned}$$

where $K^{(i)}$, $L^{(i)}$ have degree $i$ in $y_1, y_2, y_3$ and similarly we write

$$\begin{aligned} \xi &= \xi^{(0)} + \xi^{(1)} \\ \eta &= \eta^{(0)} + \eta^{(1)}. \end{aligned}$$

We begin with $K^{(2)}$, which is calculated from (14) as

$$K^{(2)} = 2f(y_1s_2 - fa)v.$$

We must find $\xi^{(1)}$ such that

$$K^{(2)} = \begin{pmatrix}1, & u, & v, & uv\end{pmatrix} B\xi^{(1)} + \text{ lower order terms.} \tag{17}$$

Comparing coefficients of $y_1^2$ and $y_3^2$, the only solution is

$$\xi_3^{(1)} = \frac{2}{\beta}(s_2 - a)y_1 - 2\alpha^2 a y_3,$$

with the other $\xi_i^{(1)} = 0$. Then the coefficient of $y_1 y_3$ in (17) dictates that

$$s_2 = (1 - \alpha\beta)a$$

and therefore $\xi_3^{(1)} = -2\alpha a f$. An exactly similar calculation with $L^{(2)}$ and $\eta_2^{(1)}$ yields

$$t_2 = (1 - \alpha\beta)d$$

and $\eta_2^{(1)} = -2\beta d g$.

Proceeding to the calculation for $K^{(1)}$, we must solve

$$K^{(1)} - \xi_3^{(1)}(t_4 u + t_5 v + t_2 uv) = \begin{pmatrix}1, & u, & v, & uv\end{pmatrix} B\xi^{(0)} + \text{ lower order terms} \tag{18}$$

where the term involving $\xi_3^{(1)}$ is necessary to account for the lower order terms from equation (17). Now examining the coefficient of $uv$ in (18), we obtain

$$2f(s_5 + \alpha a t_2) = s_2 \xi_2^{(0)} + t_2 \xi_3^{(0)}.$$

However, $\xi^{(0)}$ has degree 0 in $y_i$ by construction, so the left hand side must be identically 0. Hence

$$s_5 = -\alpha a t_2$$

and by considering the coefficient of $uv$ in $L^{(1)}$ we find

$$t_4 = -\beta d s_2.$$

Comparing coefficients of $u$ and $v$ in equation (18) we obtain

$$\begin{aligned} 6(1-\alpha\beta)a^2 df &= (f + s_4)\xi_2^{(0)} + t_4 \xi_3^{(0)} + \text{ lower order terms} \\ 2a(-s_4(f + \alpha g) + \alpha f t_5 - s_2^2 y_3) &= s_5 \xi_2^{(0)} + (g + t_5)\xi_3^{(0)} + \text{ lower order terms.} \end{aligned}$$

Since $\xi^{(0)}$ has degree 0 in $y_i$ we must have $\xi_2^{(0)} = 6(1 - \alpha\beta)a^2 d$. Moreover the coefficient of $v$ must be divisible by $g$, which is equivalent to

$$\alpha t_5 - s_4 = -\beta(1 - \alpha\beta)a^2. \tag{19}$$

By considering the coefficients of $u$, $v$ in $L^{(1)}$ in the same way we get $\eta_3^{(0)} = 6(1 - \alpha\beta)ad^2$ and a further restriction on $s_4$, $t_5$:

$$t_5 - \beta s_4 = \alpha(1 - \alpha\beta)d^2. \tag{20}$$

Solving equations (19), (20) simultaneously forces

$$\begin{aligned} s_4 &= \beta a^2 + \alpha^2 d^2 \\ t_5 &= \beta^2 a^2 + \alpha d^2, \end{aligned}$$

which in turn means that

$$\begin{aligned} \xi_3^{(0)} &= -2(1-\alpha\beta)a^3 - 2\alpha a s_4 \\ \eta_2^{(0)} &= -2(1-\alpha\beta)d^3 - 2\beta d t_5. \end{aligned}$$

We can finally write out $\xi$ and $\eta$ in full

$$\begin{aligned} \xi_2 &= 6(1-\alpha\beta)a^2 d & \eta_2 &= -2\beta d(g+t_5) - 2(1-\alpha\beta)d^3 \\ \xi_3 &= -2\alpha a(f+s_4) - 2(1-\alpha\beta)a^3 & \eta_3 &= 6(1-\alpha\beta)ad^2, \end{aligned}$$

where the other $\xi_i = \eta_i = 0$. It is necessary to check that $\xi$ and $\eta$ actually solve equations (16) when all the lower order terms are replaced, which can be verified directly.

The extended equations (12), (13) are obtained by writing out the vectors $\xi$, $\eta$ in terms of the generators of $R + Rz_1 + Rz_2$

$$\begin{aligned} z_1^2 - y_1(f+s_4)^2 &= -4(f+s_4)s_2 a y_3 - 4 s_2 s_5 d y_1 + s_2^2 y_1 y_3 + s_5^2 y_3 \\ &\quad + \xi_2 z_1 + \xi_3 z_2 \\ z_2^2 - y_3(g+t_5)^2 &= -4(g+t_5)t_2 d y_1 - 4 t_2 t_4 a y_3 + t_2^2 y_1 y_3 + t_4^2 y_1 \\ &\quad + \eta_2 z_1 + \eta_3 z_2. \end{aligned}$$

This concludes the proof of theorem (4.1).

**Proof of corollary** First observe that the fourth column of $B$ is equivalent to $y_2 = bu + cv$. Thus the extension of $Q_1$ is calculated by expressing $fy_2(bu+cv)$ in terms of the other columns of $B$. We have to find $\nu$ such that

$$(f+s_4)y_2(bu+cv) = \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\nu.$$

The solution to this linear algebra problem is

$$\begin{aligned} \nu_2 &= 2by_2 + 2(\beta ab - cd)c & \nu_3 &= 2(\alpha b^2 + c^2)a \\ \nu_4 &= -\beta a s_2 y_2 - 2ac(g+t_5) + 2(cd - \beta ab)s_5 & \nu_5 &= b(f+s_4) - \beta a b s_2 \\ \nu_6 &= -c(f+s_4) - \beta a c s_2 + 2bs_5 & \nu_7 &= 2bs_2, \end{aligned}$$

where $\nu_1 = y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7$ uses the first column of $B$ to remove any excess terms. Thus the equation extending $Q_1$ is

$$\widetilde{Q}_1\colon (f+s_4)y_2^2 = \nu_1 + \nu_2 z_1 + \nu_3 z_2.$$

Similar calculations give the equations extending $Q_2$, $Q_3$, $Q_4$ for which we list the corresponding vectors below. The equation extending $Q_2$ is

$$\widetilde{Q}_2\colon y_2^3 = \nu_1 + \nu_2 z_1 + \nu_3 z_2,$$

where

$$\begin{aligned}
\nu_1 &= y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7\\
\nu_2 &= \frac{2}{\alpha\beta - 1}(b^2 + \beta c^2)b\\
\nu_3 &= \frac{2}{\alpha\beta - 1}(\alpha b^2 + c^2)c\\
\nu_4 &= \frac{2}{1 - \alpha\beta}\left(b^2(f + s_4) + c^2(g + t_5)\right) + (\beta ac + \alpha bd)y_2 + 2(2 - \alpha\beta)abcd\\
\nu_5 &= -by_2 + 2c^2d + (\beta ac + \alpha bd)b\\
\nu_6 &= -cy_2 + 2ab^2 + (\beta ac + \alpha bd)c\\
\nu_7 &= -2bc.
\end{aligned}$$

The equation extending $Q_3$ is

$$\widetilde{Q}_3\colon (g + t_5)y_2^2 = \nu_1 + \nu_2 z_1 + \nu_3 z_2,$$

where

$$\nu_1 = y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7$$

$$\begin{aligned}
\nu_2 &= 2(b^2 + \beta c^2)d & \nu_3 &= 2cy_2 - 2(ab - \alpha cd)b\\
\nu_4 &= -\alpha dt_2y_2 - 2bd(f + s_4) + 2(ab - \alpha cd)t_4 & \nu_5 &= -b(g + t_5) - \alpha bdt_2 + 2ct_4\\
\nu_6 &= c(g + t_5) - \alpha cdt_2 & \nu_7 &= 2ct_2.
\end{aligned}$$

Finally, equation $Q_4$ is extended by

$$\widetilde{Q}_4\colon (f + s_4)(g + t_5)y_2 = \nu_1 + \nu_2 z_1 + \nu_3 z_2$$

where

$$\nu_1 = y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7$$

$$\begin{aligned}
\nu_2 &= b(g + t_5) + ct_4 - t_2y_2 & \nu_3 &= c(f + s_4) + bs_5 - s_2y_2\\
\nu_4 &= -s_5t_4 & \nu_5 &= -t_2(f + s_4) - s_2t_4\\
\nu_6 &= -s_2(g + t_5) - s_5t_2 & \nu_7 &= -2s_2t_2.
\end{aligned}$$

This completes the proof of the corollary.

Given Theorem 4.1 and its Corollary, we can prove that there is a unique hyperelliptic Fano 6-fold $W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$ extending any given projected hyperelliptic K3 surface $T'_{6,6}$. Simply take the combination of equations (12), (13) and $\widetilde{Q}_i$ corresponding to the choice (8) made in the definition of $T'_{6,6}$. This proves the Main Theorem 1.1.

# 5 Godeaux surfaces with torsion $\mathbb{Z}/2$

In this section we describe a family of hyperelliptic surfaces $Y$ of general type with $p_g = 1$, $q = 0$, and $K^2 = 2$. Each surface $Y$ is a complete intersection inside a key variety $W$ constructed in Main Theorem 1.1. We then find an appropriate subfamily of surfaces that are Galois étale $\mathbb{Z}/2$-coverings of Godeaux surfaces. In particular, we give an explicit description of the fixed point free $\mathbb{Z}/2$-action on $Y$, which is the restriction of an appropriate $\mathbb{Z}/2$-action on the key variety $W$.

## 5.1 Coverings of Godeaux surfaces

Let $X$ be the canonical model of a surface of general type with $p_g = 0$, $K^2 = 1$. We call $X$ a Godeaux surface, and we assume that the torsion subgroup $\operatorname{Tors} X \subset \operatorname{Pic} X$ has order 2. Write $\sigma$ for the generator of $\operatorname{Tors} X$, and consider the Galois étale double covering $f\colon Y \to X$ induced by $\sigma$. The covering surface is constructed by taking $Y = \operatorname{Proj} R(Y, K_X, \sigma)$, or written out in full

$$Y = \operatorname{Proj} \bigoplus_{n\geq 0} \left(H^0(X, nK_X) \oplus H^0(X, nK_X + \sigma)\right).$$

The surface $Y$ is the canonical model of a surface of general type with $p_g = 1$, $q = 0$, $K^2 = 2$, and the extra $\mathbb{Z}/2$-grading on the ring $R(Y, K_X, \sigma)$ determines a fixed point free $\mathbb{Z}/2$ group action on $Y$, where the first summand is invariant and the second is anti-invariant. The quotient by this group action is the map $f$.

Moreover, an analysis of $R(Y, K_X, \sigma)$ reveals that the canonical curve of $Y$ must be hyperelliptic.

**Lemma 5.1** *If $Y$ is the unramified double covering of a Godeaux surface with torsion $\mathbb{Z}/2$, then the canonical curve section $D$ in $|K_Y|$ is hyperelliptic.*

This lemma was also proved in [CD], using a monomial counting proof. We use a Hilbert series approach which has some advantages, and yields slightly more information about the group action for use later.

**Proof** Define the bigraded Hilbert series of the ring $R(Y, K_X, \sigma)$ by

$$P_Y(t, e) = \sum_{n\geq 0} \left(h^0(X, nK_X)t^n + h^0(X, nK_X + \sigma)t^n e\right),$$

where $t$ keeps track of the degree, and $e$ keeps track of the eigenspace, so that $e^2 = 1$. Then using the Riemann–Roch theorem,

$$P_Y(t, e) = 1 + et + 2t^2 + 2t^2e + 4t^3 + 4t^3e + \dots$$

which can be written as the rational function

$$P_Y(t,e) = \frac{1 + (e-1)t^4 + (-2e-2)t^5 + (-4e-6)t^6 + (7e+8)t^8 + \dots}{(1-et)(1-t^2)(1-et^2)^2(1-t^3)^2(1-et^3)^2}.$$

Using well-known Hilbert series properties would normally indicate that $Y$ is a subvariety of $\mathbb{P}(1, 2^3, 3^4)$. However, the first nontrivial coefficient in the numerator is not negative, due to the bigrading. Thus we must introduce an extra generator of degree 4 in the negative eigenspace, dividing $P_Y(t,e)$ by $(1-et^4)$ so that the numerator becomes

$$1 - t^4 + (-2e-2)t^5 + (-4e-6)t^6 + \dots.$$

The extra $-t^4$ term in the numerator suggests that it is necessary to introduce a relation in degree 4, which does not eliminate the new generator of degree 4. Thus the canonical curve section of $Y$ must be hyperelliptic, and is given by $D \subset \mathbb{P}(2^4, 3^4, 4)$.

Now, we can construct hyperelliptic surfaces $Y$ of general type using the key variety of Main Theorem 1.1.

**Theorem 5.2** *There is a* 15*-parameter family of hyperelliptic surfaces* $Y$ *of general type with* $p_g = 1$, $q = 0$, $K^2 = 2$ *and no torsion, each of which is a complete intersection of type* $(1,1,1,2)$ *in a Fano* 6*-fold* $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ *with* $10 \times \frac{1}{2}$ *points.*

The proof is identical to that of [Co2], Theorem 4.1, and we obtain the canonical model of $Y$ using this construction. In order to find Godeaux surfaces with $\mathbb{Z}/2$-torsion, we must determine which hyperelliptic surfaces $Y$ have an appropriate $\mathbb{Z}/2$-action. To do this we study the $\mathbb{Z}/2$-action on the hyperelliptic curve $D$, and extend it to the key variety $W$.

## 5.2 The canonical curve

Now let $f\colon Y \to X$ be the étale double cover of a Godeaux surface $X$ and suppose $D$ is a nonsingular curve in $|K_Y|$, similarly $C$ in $|K_X + \sigma|$. By Lemma 5.1, $D$ is a hyperelliptic curve and $C$ has genus 2 so is automatically hyperelliptic too. Let $\pi_D\colon D \to Q \cong \mathbb{P}^1$ denote the quotient map of the hyperelliptic involution on $D$, similarly $\pi_C\colon C \to \mathbb{P}^1$. Since $D$ is an unramified double cover of $C$ via $f|_D$, this induces a double covering of $\operatorname{Im}\pi_C = \mathbb{P}^1$ by $Q$.

We get the following picture:

$$\begin{tikzcd} E \arrow[r] & D \arrow[r, "f|_D"] \arrow[d, "\pi_D"] & C \arrow[d, "\pi_C"] \\ & Q \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

There is a fixed point free involution on the curve $D$ induced by the unramified double cover $f|_D$, which we call the Godeaux involution. We use the same notation for the Godeaux involution and the torsion element $\sigma \in \operatorname{Pic} X$. The Weierstrass points of $D$ must be invariant under $\sigma$, so there is a natural division of these eight points into two sets $\{P_1, \dots, P_4\}$ and $\{P_5, \dots, P_8\}$, which are interchanged by $\sigma$.

Now consider the ineffective theta characteristic $A_D = K_Y|_D$ on $D$ which is determined by the surface $Y$. The divisor class $A_D$ is invariant under both $\sigma$ and the hyperelliptic involution, so a priori the only possibilities are

$$\begin{aligned} A_D &\sim P_1 + P_2 + P_3 + P_4 - g^1_2 \sim P_5 + P_6 + P_7 + P_8 - g^1_2, \\ A_D &\sim P_1 + P_3 + P_5 + P_7 - g^1_2 \sim P_2 + P_4 + P_6 + P_8 - g^1_2. \end{aligned} \tag{21}$$

The difference between these is that the former is only $\sigma$-invariant as a divisor class, whereas the latter is an $\sigma$-invariant divisor.

Now, we have already constructed the graded ring $R(D, A_D)$ in section 2.1. Furthermore by the adjunction formula, it is clear that $2g^1_2 \sim 2A_D \sim K_D$. However, these two divisor classes $A_D$ and $g^1_2$ are distinct, because the $g^1_2$ is effective whereas $A_D$ is ineffective. Thus we have a 2-torsion class

$$\tau = A_D - g^1_2$$

on $D$, which corresponds to a genus 5 unramified double cover $E$ of $D$, where

$$E = \operatorname{Proj} R(D, A_D, \tau) = \operatorname{Proj} \bigoplus_{n \geq 0} \left( H^0(D, nA_D) \oplus H^0(D, nA_D + \tau) \right).$$

We outline the procedure to construct the bigraded ring $R(D, A_D, \tau)$. Using the notation of section 2, write $s_1, s_2$ for the sections of the $g^1_2$ and

$$u \colon \mathcal{O}_D \to \mathcal{O}_D(P_1 + \cdots + P_4), \quad v \colon \mathcal{O}_D \to \mathcal{O}_D(P_5 + \cdots + P_8).$$

We can very quickly write down generators and relations for $R(D, A_D, \tau)$:

| $n$ | $H^0(D, nA_D)$ | $H^0(D, nA_D + \tau)$ |
|---|---|---|
| 0 | $k$ | $\phi$ |
| 1 | $\phi$ | $s_1, s_2$ |
| 2 | $s_1^2, s_1 s_2, s_2^2$ | $u, v$ |
| 3 | $\dots$ | $\dots$ |

Thus $E$ is a complete intersection

$$E_{4,4} \subset \mathbb{P}(1, 1, 2, 2),$$

defined by equations $u^2 = f_4(s_1, s_2)$ and $v^2 = g_4(s_1, s_2)$. The polynomials $f$ and $g$ are functions on $\mathbb{P}^1$ whose vanishing determines the splitting of the Weierstrass points of $D$ into two sets of four.

The curve $E$ comes bundled at no extra cost with the fixed point free involution $\tau\colon E \to E$ associated to the torsion $\tau$ of $D$. We recover the restricted algebra $R(D, K_Y|_D)$ of section 2.1 by taking the $\tau$-invariant subring of $R(D, A_D, \tau)$:

$$R(D, A_D) = R(D, A_D, \tau)^{\langle\tau\rangle}.$$

For future reference, we write out the action of $\tau$ on $E$ using the eigenspace table above

$$s_1 \mapsto -s_1, \quad s_2 \mapsto -s_2, \quad u \mapsto -u, \quad v \mapsto -v.$$

Now, we claim that the covering curve $E$ completely determines the Godeaux involution $\sigma$ on $D$. First observe that $D$ is a quotient of $E$, and that this covering curve only exists because $D$ is the curve section of $|K_Y|$. Thus $\sigma$ lifts to the curve $E$ and should be compatible with the involution $\tau$ on $E$, so that $\sigma^2 = 1$ or $\tau$ on $E$.

**Proposition 5.3** *The action of $\sigma$ on $E$ is given by*

$$s_1 \mapsto is_1, \quad s_2 \mapsto -is_2, \quad u \mapsto iv, \quad v \mapsto iu,$$

*so that $\sigma^2 = \tau$ and the group $\langle\sigma, \tau\rangle$ is isomorphic to $\mathbb{Z}/4$. Moreover, the polarising divisor of $D$ is*

$$A_D \sim P_1 + P_2 + P_3 + P_4 - g_2^1 \sim P_5 + P_6 + P_7 + P_8 - g_2^1.$$

**Proof** The Hilbert series of Lemma 5.1 gives the eigenspace decomposition of $\sigma$ on $D$, which we must abide by. In particular, $R(D, A_D)$ should have only one invariant generator in degree 2, and the generator in degree 4 should be anti-invariant. This forces $\sigma^2 = \tau$, so that the group $\langle\sigma, \tau\rangle$ acting on $E$ is $\mathbb{Z}/4$ rather than $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Now, there are two possibilities for $\sigma$ depending on the representation of $A_D$ chosen from equation (21). The correct choice is

$$s_1 \mapsto is_1, \quad s_2 \mapsto -is_2, \quad u \mapsto iv, \quad v \mapsto iu,$$

which corresponds to the first possibility in (21). Indeed, the alternative

$$s_1 \mapsto is_1, \quad s_2 \mapsto -is_2, \quad u \mapsto iu, \quad v \mapsto iv,$$

is obliged to have two fixed points on $D$ at the coordinate points $P_{y_1}$ and $P_{y_3}$, and so can not possibly be the Godeaux involution.

Hence we have

$$A_D \sim P_1 + P_2 + P_3 + P_4 - g_2^1 \sim P_5 + P_6 + P_7 + P_8 - g_2^1,$$

with corresponding action on $R(D, A_D)$ given by

$$\operatorname{rank}\left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & f_2 & t \\ z_3 & z_4 & t & g_2 \end{array}\right) \le 1 \quad \mapsto \quad \operatorname{rank}\left(\begin{array}{cc|cc} -y_1 & y_2 & -z_3 & -z_1 \\ y_2 & -y_3 & z_4 & z_2 \\ \hline -z_3 & z_4 & -g_2 & -t \\ -z_1 & z_2 & -t & -f_2 \end{array}\right) \le 1,$$

where

$$\begin{aligned} f_2 &= \alpha_1 y_1^2 + \alpha_2 y_1 y_2 + \alpha_3 y_1 y_3 + \alpha_4 y_2^2 + \alpha_5 y_2 y_3 + \alpha_6 y_3^2 \\ g_2 &= -\alpha_1 y_1^2 + \alpha_2 y_1 y_2 - \alpha_3 y_1 y_3 - \alpha_4 y_2^2 + \alpha_5 y_2 y_3 - \alpha_6 y_3^2 \end{aligned}$$

and the involution has no fixed points as long as $\alpha_1$ and $\alpha_6$ are not zero. This proves the proposition.

## 5.3 Involution on the K3 surface

Moving one step up the tower, we lift the involution $\sigma$ on the canonical curve section $D$ to the hyperelliptic K3 surface $T \subset \mathbb{P}(2^4, 3^4, 4)$, which contains $D$ as a quadric section.

The whole argument becomes quite transparent when viewed in terms of commutative algebra. The graded ring $R(T, A_T)$ is described explicitly in section 3.1, and after eliminating one of the generators in degree 2, we obtain $R(D, A_D)$. Now if $D$ is the unramified double covering of a Godeaux curve $C$ with its involution $\sigma\colon D \to D$ from Proposition 5.3, then:

**Proposition 5.4** *There is at least one K3 surface $T$ containing the curve $D$ such that the involution $\sigma$ on $D$ has a unique lift to $T$. Moreover, such a lift $\sigma\colon T \to T$ has four fixed points which are $\frac{1}{2}$ points of $T$. We call $\sigma$ the Godeaux involution on $T$.*

**Remark 5.5** This is surprising because we are looking for a fixed point free involution on the covering surface $Y$, so it would be reasonable to expect that the involution on the K3 surface is free.

**Proof** *Step (1) Determining the character of $\sigma$.* Temporarily choose coordinates on $T$ so that $D = T \cap (y_4 = 0)$, where $y_4$ must be semi-invariant under any putative involution. Then the determinantal equations (1), which partially define $T$, take the general form

$$\operatorname{rank}\begin{pmatrix} y_1 + \alpha y_4 & y_2 + \beta y_4 & z_1 \\ y_2 + \gamma y_4 & y_3 + \delta y_4 & z_2 \\ z_3 & z_4 & t \end{pmatrix} \le 1,$$

where $\alpha, \beta, \gamma, \delta$ are scalars. Now if $\sigma$ lifts to $T$, then our choice of coordinates means that the action of $\sigma$ on $T$ is predetermined by $\sigma|_D$ from Proposition 5.3, excepting the new variable $y_4$. Since $T$ is a double covering of a quadric $Q \subset \mathbb{P}^3$ of rank 4, the determinantal equations force $\alpha = \delta$ and $\beta = -\gamma$. Thus $y_4$ is anti-invariant, and the signature of $\sigma$ on $Q$ is $(1, 3)$. We can recalibrate the coordinate system so that the determinantal equations and involution on $T$ are

$$\operatorname{rank}\left(\begin{array}{cc|c} y_1 & y_2 & z_1 \\ y_3 & y_4 & z_2 \\ \hline z_3 & z_4 & t \end{array}\right) \leq 1 \quad \mapsto \quad \operatorname{rank}\left(\begin{array}{cc|c} -y_1 & y_3 & -z_3 \\ y_2 & -y_4 & z_4 \\ \hline -z_1 & z_2 & -t \end{array}\right) \leq 1,$$

where the original curve $D$ is obtained from $T$ by taking the anti-invariant quadric section $y_2 = y_3$.

*Step (2) Fixed points of $\sigma$.* First observe that the involution on $T$ swaps the two branch curves, and also swaps the sheets of the hyperelliptic double covering $\pi\colon T \to Q$. Thus any fixed points of $\sigma$ lie on both components of the branch curve, and so must be $\frac{1}{2}$ points of $T$.

For a $\frac{1}{2}$ point of $T$ to be fixed under $\sigma$, one of two things must happen:

$$y_1 = y_4 = y_2 - y_3 = 0, \text{ or } y_2 + y_3 = 0.$$

The only case we need to worry about is when $y_2 + y_3 = 0$ since the other case reduces to the curve $D$, on which $\sigma$ is fixed point free by hypothesis. To ensure $C_1$ and $C_2$ are interchanged under $\sigma$, their equations are of the form

$$\begin{aligned} f_{3,1} &= \alpha_1 s_1^3 t_1 + \alpha_2 s_1^2 s_2 t_1 + \alpha_3 s_1 s_2^2 t_1 + \alpha_4 s_2^3 t_1 \\ &\quad + \beta_1 s_1^3 t_2 + \beta_2 s_1^2 s_2 t_2 + \beta_3 s_1 s_2^2 t_2 + \beta_4 s_2^3 t_2, \\ g_{1,3} &= -\alpha_1 s_1 t_1^3 + \alpha_2 s_1 t_1^2 t_2 - \alpha_3 s_1 t_1 t_2^2 + \alpha_4 s_1 t_2^3 \\ &\quad + \beta_1 s_2 t_1^3 - \beta_2 s_2 t_1^2 t_2 + \beta_3 s_2 t_1 t_2^2 - \beta_4 s_2 t_2^3, \end{aligned} \tag{22}$$

on $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Note that there is more than one possible choice of $f_{3,1}$, $g_{1,3}$ for which $T \cap (y_2 = y_3) = D$, so we can not claim that $T$ is unique in the statement of the proposition. We can assume $y_2 = 1$, $y_3 = -1$, then for a general choice of branch curve there are four fixed points on $T$. These are $(\lambda, 1, -1, -1/\lambda, 0, 0, 0, 0, 0)$, where $\lambda$ is a root of the quartic equation derived from evaluating equations (22)

$$\alpha_1 \lambda^4 + (\alpha_2 - \beta_1)\lambda^3 + (\alpha_3 - \beta_2)\lambda^2 + (\alpha_4 - \beta_3)\lambda - \beta_4,$$

which proves the proposition.

## 5.4 Involution on the Fano 6-fold

We extend the involution on the K3 surface $T$ to the Fano 6-fold $W$ constructed in Main Theorem 1.1. To do this we use a $\mathbb{Z}/2$-equivariant form of the projection–unprojection construction described in Section 3.3.

We begin with a $\mathbb{Z}/2$-equivariant unprojection construction for the K3 surface $T$ with a Godeaux involution. Recall from Proposition 5.4 that if $T$ has a Godeaux involution $\sigma$, then $\sigma$ has four fixed points, each of which is a $\frac{1}{2}$ point. Let $P$ be one of these fixed points, and project from $P$, to obtain

$$\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2,2,2,3,3).$$

Here $T'$ is a double cover of $\mathbb{P}^2$ branched in two nodal cubics, and $P$ is projected to the image of $\varphi$, a double covering of the line through the two nodes. There is an induced involution on $T'$, which swaps the two branch cubics and leaves the image of $\varphi$ invariant. We call this a $\mathbb{Z}/2$-*equivariant projection.*

Examining the equations of $T'$ as described in Section 3.3, the determinantal equations for $T$ are

$$\operatorname{rank}\begin{pmatrix} y_2 & f & z_1 \\ g & y_4 & z_3 \\ z_2 & z_4 & t \end{pmatrix} \leq 1,$$

where $f = y_1 + \alpha y_3$, $g = \alpha y_1 + y_3$ because the two branch curves are interchanged by $\sigma$. Proposition 5.4 fixes the involution on $T$ as

$$\begin{gathered} f \mapsto g, \quad y_2 \mapsto -y_2, \quad g \mapsto f, \quad z_1 \mapsto -z_2, \quad z_2 \mapsto -z_1, \\ y_4 \mapsto -y_4, \quad z_3 \mapsto z_4, \quad z_4 \mapsto z_3, \quad t \mapsto -t \end{gathered} \tag{23}$$

and we note that this implies $\sigma(y_1) = y_3$, $\sigma(y_3) = y_1$. Hence referring to Section 3.3 the equations of $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ must be of the form

$$\begin{aligned} z_1^2 &= y_1 f^2 + y_2^2(l_1 f + l_2 y_2 + l_3 g) + l_4 y_2 f g \\ z_2^2 &= y_3 g^2 + y_2^2(l_3 f - l_2 y_2 + l_1 g) - l_4 y_2 f g, \end{aligned} \tag{24}$$

where $l_i$ are scalars. The remaining equations of $T$ can be calculated from those of $T'$ using the unprojection procedure outlined in Section 3.3.

There are three isolated fixed points on $T'$ when $z_1 = z_2 = y_1 + y_3 = 0$, which correspond to three of the $9 \times \frac{1}{2}$ points as expected. Further, $T'$ has two fixed points on the unprojection divisor which arise from the fact that the centre of projection $P$ was itself a fixed point. Indeed, suppose we have a local orbifold chart for a neighbourhood of the $\frac{1}{2}$ point $P$ in $T$. This is the quotient of $\mathbb{C}^2$ by $\mathbb{Z}/2$ acting by $-1$ on both coordinates. Then writing $u, v$ for the coordinates on $\mathbb{C}^2$, $\sigma$ lifts to the chart as

$$u \mapsto -iv, \quad v \mapsto -iu$$

by equation (23). The Kawamata $(1,1)$ weighted blowup at $P$ introduces the ratio $(u : v)$ as the exceptional $\mathbb{P}^1$, which is then embedded in $T'$ by $\varphi$. Thus the induced action of $\sigma$ on the image of $\varphi$ inside $T'$ has two fixed points at $\varphi(1,1)$ and $\varphi(-1,1)$.

It is important to note that the $\mathbb{Z}/2$-equivariant projection–unprojection construction for $T$ relies on the choice of $\frac{1}{2}$ point $P$. As such we can no longer assume there is a canonical choice of curve $D \subset T$ defined by setting $f = g$, as we did in the proof of Proposition 5.4. The choice of covering curve $D$ is made by taking any anti-invariant quadric section of $T$ which avoids the $10 \times \frac{1}{2}$ points. Indeed, the quadric $f = g$ contains the point $P$ and so is no longer a valid choice.

Now, we claim that the involution on $T$ can be extended to the Fano 6-fold $W$ at the top of the tower.

**Proposition 5.6** *Suppose $T \subset \mathbb{P}(2^4, 3^4, 4)$ is a K3 surface with $10\times\frac{1}{2}$ points and $\sigma\colon T \to T$ is a Godeaux involution lifted from some quadric section $D \subset T$. Then there is a lift of $\sigma$ to the unique Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ extending $T$ which was constructed in Main Theorem 1.1. Moreover the fixed locus of the involution $\sigma\colon W \to W$ consists of four isolated $\frac{1}{2}$ points.*

**Proof** Project from one of the fixed $\frac{1}{2}$ points on $T$ to get

$$\varphi\colon \mathbb{P}^1 \to T'_{6,6} \subset \mathbb{P}(2^3, 3^2).$$

Following the extension procedure outlined in the proof of Main Theorem 1.1, the extended map

$$\Phi\colon \mathbb{P}^5 \to W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$$

must be

$$\Phi\colon (a, b, c, d, u, v) \mapsto (a, b, c, d, u^2 + 2av, bu + cv, v^2 + 2du, f_1, f_2),$$

where

$$\begin{aligned} f_1 &= u\left(f + \alpha(a^2 + \alpha d^2)\right) + (1-\alpha^2)auv + \alpha(\alpha^2 - 1)adv, \\ f_2 &= v\left(g + \alpha(\alpha a^2 + d^2)\right) + (1-\alpha^2)duv + \alpha(\alpha^2 - 1)adu. \end{aligned}$$

To make $\Phi$ compatible with the lift of $\sigma\colon T \to T$ defined by equation (23), the action on $\mathbb{P}^5$ must be

$$u \mapsto -v, \quad v \mapsto -u, \quad a \mapsto -d, \quad b \mapsto c, \quad c \mapsto b, \quad d \mapsto -a.$$

Thus $\Phi$ is $\sigma$-equivariant, so the equations defining the image of $\Phi$ are invariant and consequently $W' \subset \mathbb{P}(1^4, 2^3, 3^2)$ can be chosen to be invariant. Alternatively, a direct calculation following the proof of Main Theorem 1.1

demonstrates explicitly that the equations of the image of $\Phi$ are invariant. Hence by $\mathbb{Z}/2$-equivariant unprojection, the involution lifts to the 6-fold $W$.

Now outside the image of $\Phi$, there are just three isolated points on $W'$ that are fixed under $\sigma$. These are the same $\frac{1}{2}$ points that were fixed under $\sigma|_{T'}$. On the image of $\Phi$ itself there are two copies of $\mathbb{P}^2 \subset \mathbb{P}^5$ whose image under $\Phi$ are fixed by $\sigma$. These are defined by

$$\mathbb{P}^5 \cap (u = v, a = d, b = -c), \quad \mathbb{P}^5 \cap (u = -v, a = -d, b = c),$$

and they are the analogue of the two fixed points on $\varphi(\mathbb{P}^1) \subset T'$. Fortunately these nonisolated fixed loci are contracted to the centre of projection $P$ on $W$, so that $\sigma$ fixes just four isolated $\frac{1}{2}$ points there. This proves the proposition.

## 6 Godeaux surfaces with torsion $\mathbb{Z}/2$

Given a hyperelliptic tower $D \subset T \subset W$ where $W$ is the unique Fano 6-fold extending the K3 surface $T$, suppose the curve $D$ is a double cover of a Godeaux curve $C$. Now, suppose further that the tower is constructed so that the Godeaux involution $\sigma$ on $D$ lifts to $T$ and subsequently $W$ as described in Propositions 5.4 and 5.6. Write $A$ for the hyperplane class on $W$ so that $\mathcal{O}_W(A) = \mathcal{O}_W(1)$, and $-K_W = 4A$. Then $\sigma$ induces a $\mathbb{Z} \oplus \mathbb{Z}/2$-bigrading on the ring $R(W, A)$ according to eigenspace:

| $n$ | $H^0(W, nA)^+$ | $H^0(W, nA)^-$ |
|---|---|---|
| 1 | $a - d, b + c$ | $a + d, b - c$ |
| 2 | $y_1 + y_3$ | $y_1 - y_3, y_2, y_4$ |
| 3 | $z_1 - z_2, z_3 + z_4$ | $z_1 + z_2, z_3 - z_4$ |
| 4 | | $t$ |

Now by Theorem 5.2, we can construct a surface $Y$ of general type with $p_g = 1$, $K^2 = 2$ as a complete intersection inside $W$ as long as $Y$ avoids the $\frac{1}{2}$ points of $W$, which is an open condition. Referring to the above table and the eigenspace decomposition on $Y$ given by lemma 5.1, if we $Y$ to be a complete intersection of type $(1^+, 1^+, 1^-, 2^-)$ inside $W$ then $\sigma|_Y$ will be the fixed point free Godeaux involution. Hence we have:

**Theorem 6.1** *There is an 8 parameter family of Godeaux surfaces with $\mathbb{Z}/2$-torsion.*

The parameter count is a matter of calculating the moduli of $W$ using Main Theorem 1.1, Section 5.4 and then counting the number of free parameters involved in choosing the complete intersection $(1^+, 1^+, 1^-, 2^-)$.

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